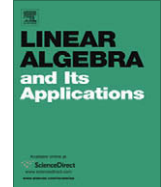


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Complex bimatrix variate generalised beta distributions

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ABSTRACT

In this paper, the study of bivariate generalised beta types I and II distributions is extended to the complex matrix variate case, for which the corresponding density functions are found. In addition, for complex bimatrix variate beta type I distributions, several basic properties, including the joint eigenvalue density and the maximum eigenvalue distribution, are studied.

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1. Introduction

Complex matrix variate distributions play an important role in various fields of research. Applications of complex random matrices can be found in multiple time series analysis, nuclear physics, complex multivariate linear model, shape theory, the evaluation of the capacity of multiple-input multiple-output (MIMO) wireless communication systems, see Mehta [23], Khatri [17], Micheas et al. [24], Ratnarajah et al. [27] among many others.

Several studies on the distribution of complex random matrices have been made, see Provost [22]. The complex matrix variate Gaussian distribution was introduced by Wooding [31], and further developed by Turin [30], and Goodman [14]. The complex Wishart distribution was studied by Goodman

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[14] and James [16], among many others. James [16] and Khatri [17] derived both complex central and noncentral matrix variate beta distributions.

When the goal is to generalise the distribution of a random variable to the multivariate case, two options are normally addressed, by which it is extended to either the vectorial or the matrix case, e.g. the normal, t or Bessel distributions, among many others. However, some of these generalisations have traditionally been made directly to the matrix case, when such a matrix is symmetric, as in the case of the chi-square and beta distributions, for which the corresponding multivariate distributions are the Wishart and matrix variate beta distributions, respectively. Nevertheless, these latter generalisations are inappropriate in some cases, because, in some applications, the researcher is interested in a vector variate, not in a symmetric matrix, see Libby and Novick [20]. In other words, the researcher is interested in a vector, say, $\mathbf{X} = (X_1, \dots, X_m)'$, such that X_i has a marginal beta type I or II distribution for all $i = 1, \dots, m$. In this respect, Liddy and Novick [20] and Chen and Novick [2] proposed a multivariate vector-version of the beta types I and II distributions. Let us consider the following bivariate version, see Olkin and Liu [26].

Let X_0, X_1, X_2 be distributed as independent gamma random variates with shape parameters $a = a_0, a_1, a_2$, respectively (see Definition 1 in Section 2); and define

$$U_1 = \frac{X_1}{X_1 + X_0}, \quad U_2 = \frac{X_2}{X_2 + X_0}. \quad (1)$$

Clearly, U_1 and U_2 each have a beta type I distribution, $U_1 \sim BI_1(a_1, a_0)$ and $U_2 \sim BI_1(a_2, a_0)$, over $0 \leq u_1, u_2 \leq 1$ (see Section 2.1). However, they are correlated such that $(U_1, U_2)'$ has a bivariate generalised beta type I distribution over $0 \leq u_1, u_2 \leq 1$. The kernel of the joint density function of U_1 and U_2 is

$$\propto \frac{u_1^{a_1-1} u_2^{a_2-1} (1-u_1)^{a_2+a_0-1} (1-u_2)^{a_1+a_0-1}}{(1-u_1 u_2)^{a_1+a_2+a_0}}, \quad 0 \leq u_1, u_2 \leq 1.$$

A similar result is obtained in the case of the beta type II. In this case, define

$$F_1 = \frac{X_1}{X_0}, \quad F_2 = \frac{X_2}{X_0}.$$

Once again it is evident that F_1 and F_2 each have a beta type II distribution, $F_1 \sim BII_1(a_1, a_0)$ and $F_2 \sim BII_1(a_2, a_0)$, over $f_1, f_2 \geq 0$. As in the beta type I case, they are correlated such that $(F_1, F_2)'$ has a bivariate generalised beta type II distribution over $f_1, f_2 \geq 0$. The kernel of the joint density function of F_1 and F_2 is

$$\propto \frac{f_1^{a_1-1} f_2^{a_2-1}}{(1+f_1+f_2)^{a_1+a_2+a_0}}, \quad f_1, f_2 \geq 0.$$

Some applications to utility modelling and Bayesian analysis are presented in Libby and Novick [20] and Chen and Novick [2], respectively.

These ideas can be extended to the matrix variate case. Thus, let us assume a partitioned matrix

$\mathbb{U} = (\mathbf{U}_1; \mathbf{U}_2)' \in \mathbb{C}^{2m \times m}$, then under the complex matrix variate versions of the transformations (1), we are interested in finding the joint density of \mathbf{U}_1 and \mathbf{U}_2 , from where it is easy to see that the marginal densities of \mathbf{U}_1 and \mathbf{U}_2 are complex matrix variate beta type I distributions. In the central and noncentral real cases, the matrix variate joint densities of \mathbf{U}_1 and \mathbf{U}_2 and of \mathbf{F}_1 and \mathbf{F}_2 , together with some of their properties, are studied in Díaz-García and Gutiérrez-Jáimez [10,11]. These distributions are termed central complex bimatrix variate generalised beta type I and II distributions, respectively.

In the present paper, the bivariate generalised beta types I and II distributions are extended to the complex matrix variate case, see Sections 3 and 4. In Section 5, certain basic properties, the joint eigenvalue density and the density of their maximum eigenvalues are studied for the complex bimatrix variate generalised beta type I distribution.

2. Preliminary results

The joint eigenvalues density can be calculated using complex hypergeometric functions and complex invariant polynomials with matrix arguments. In this section, these and other issues are addressed for the case of the complex multivariate distribution. Let us first establish some notation.

2.1. Notation and matrix variate distributions

Let $\mathbf{A} = (a_{rs}) = (a_{rs_1} + ia_{rs_2}) = \mathbf{A}_1 + i\mathbf{A}_2$ be an $m \times n$ array of complex numbers, $\mathbf{A} \in \mathbb{C}^{m \times n}$, where $\mathbf{A}_1 = (a_{rs_1})$ and $\mathbf{A}_2 = (a_{rs_2})$ are real matrices $m \times n$, $\mathbf{A}_1, \mathbf{A}_2 \in \mathbb{R}^{m \times n}$, and $i = \sqrt{-1}$. Then, \mathbf{A}' denotes the transpose of \mathbf{A} , $\bar{\mathbf{A}}$ denotes the conjugate of \mathbf{A} , and \mathbf{A}^H denotes the conjugate transpose of \mathbf{A} . For $n = m$, let $\text{tr}(\mathbf{A}) = a_{11} + \cdots + a_{mm}$, $\text{etr}(\mathbf{A}) = \exp(\text{tr}(\mathbf{A}))$, and then $|\mathbf{A}|$ denotes the determinant of \mathbf{A} . $\mathbf{A} = \mathbf{A}^H > \mathbf{0}$ is a Hermitian positive definite matrix, and $\mathbf{A}^{1/2}$ denotes the unique Hermitian positive definite square root matrix of $\mathbf{A} = \mathbf{A}^H > \mathbf{0}$. $d\mathbf{A}$ denotes the differential matrix of \mathbf{A} and $(d\mathbf{A}) = (d\mathbf{A}_1)(d\mathbf{A}_2)$ denotes the volume element (Lebesgue measure) commonly associated with \mathbf{A} . For example if \mathbf{A} is a Hermitian matrix, then \mathbf{A}_1 is a symmetric matrix ($\mathbf{A} = \mathbf{A}'$) and \mathbf{A}_2 is a skew-symmetric matrix ($\mathbf{A} = -\mathbf{A}'$). In this case

$$(d\mathbf{A}) = \bigwedge_{r \leq s} da_{rs_1} \bigwedge_{r < s} da_{rs_2}.$$

The space of all matrices $\mathbf{G}_1 \in \mathbb{C}^{n \times m}$ ($m \leq n$) with orthonormal columns is termed the **Stiefel manifold**, denoted by $\mathbb{C}\mathcal{V}_{m,n}$. Thus

$$\mathbb{C}\mathcal{V}_{m,n} = \{\mathbf{G}_1 \in \mathbb{C}^{n \times m} | \mathbf{G}_1^H \mathbf{G}_1 = \mathbf{I}_m\}.$$

From James [16]

$$\text{Vol}(\mathbb{C}\mathcal{V}_{m,n}) = \int_{\mathbf{G}_1 \in \mathbb{C}\mathcal{V}_{m,n}} (\mathbf{G}_1^H d\mathbf{G}_1) = \frac{2^m \pi^{mn}}{\mathbb{C}\Gamma_m[n]},$$

where $\mathbb{C}\Gamma_m[a]$ denotes the complex multivariate gamma function and is defined as

$$\mathbb{C}\Gamma_m[a] = \int_{\mathbf{V}=\mathbf{V}^H > \mathbf{0}} \text{etr}(-\mathbf{V}) |\mathbf{V}|^{a-m} (d\mathbf{V}) = \pi^{m(m-1)/2} \prod_{j=1}^m \Gamma[a-j+1],$$

where $\text{Re}(a) > m-1$.

If $m = n$, we have a special case of the Stiefel manifold, termed unitary manifold or unitary group and denoted as $\mathcal{U}(m) \equiv \mathbb{C}\mathcal{V}_{m,m}$.

Definition 1 (The complex multivariate beta function). The complex multivariate beta function, denoted as $\mathbb{C}\beta_m[a, b]$, is defined by

$$\begin{aligned} \mathbb{C}\beta_m[b, a] &= \int_{\mathbf{0} < \mathbf{S} = \mathbf{S}^H < \mathbf{I}_m} |\mathbf{S}|^{a-m} |\mathbf{I}_m - \mathbf{S}|^{b-m} (d\mathbf{S}) \\ &= \int_{\mathbf{R} = \mathbf{R}^H > \mathbf{0}} |\mathbf{R}|^{a-m} |\mathbf{I}_m + \mathbf{R}|^{-(a+b)} (d\mathbf{R}), \quad \mathbf{R} = (\mathbf{I} - \mathbf{S})^{-1} - \mathbf{I} \\ &= \frac{\mathbb{C}\Gamma_m[a] \mathbb{C}\Gamma_m[b]}{\mathbb{C}\Gamma_m[a+b]}, \end{aligned}$$

where $\text{Re}(a) > m-1$ and $\text{Re}(b) > m-1$.

We now give definitions of the complex matrix variate gamma, beta type I and II distributions, see James [16], Khatri [17], Mathai [21].

Definition 2 (Complex matrix variate gamma distribution). It is said that $\mathbf{A} \in \mathbb{C}^{m \times m}$, a random Hermitian positive definite matrix, has a complex matrix variate gamma distribution with parameters a and a Hermitian positive definite matrix $\mathbf{\Theta} \in \mathbb{C}^{m \times m}$, if its density function is

$$\frac{1}{\mathfrak{C}\Gamma_m[a]|\mathbf{\Theta}|^a} |\mathbf{A}|^{a-m} \text{etr}(-\mathbf{\Theta}^{-1}\mathbf{A})(d\mathbf{A}), \quad \mathbf{A} = \mathbf{A}^H > \mathbf{0}, \quad (2)$$

where $\text{Re}(a) > m - 1$. Such a distribution is denoted as $\mathbf{A} \sim \mathfrak{C}\mathcal{G}_m(a, \mathbf{\Theta})$.

Lemma 1 (Complex matrix variate beta type I and II distribution). If \mathbf{A} and \mathbf{B} have a complex matrix variate gamma distributions, i.e. $\mathbf{A} \sim \mathfrak{C}\mathcal{G}_m(a, \mathbf{I}_m)$ and $\mathbf{B} \sim \mathfrak{C}\mathcal{G}_m(b, \mathbf{I}_m)$ independently.

1. Then the complex matrix variate beta type I distribution is defined as

$$\mathbf{U} = \begin{cases} (\mathbf{A} + \mathbf{B})^{-1/2} \mathbf{A} (\mathbf{A} + \mathbf{B})^{-1/2}', & \text{Definition 1 or,} \\ \mathbf{A}^{1/2} (\mathbf{A} + \mathbf{B})^{-1} (\mathbf{A}^{1/2})', & \text{Definition 2.} \end{cases} \quad (3)$$

Thus under Definitions 1 and 2 its density function is denoted as

$$\mathfrak{C}\mathcal{B}I_m(\mathbf{U}; a, b),$$

and given by

$$\frac{1}{\mathfrak{C}\beta_m[a, b]} |\mathbf{U}|^{a-m} |\mathbf{I}_m - \mathbf{U}|^{b-m} (d\mathbf{U}), \quad \mathbf{0} < \mathbf{U} = \mathbf{U}^H < \mathbf{I}_m, \quad (4)$$

this being denoted as $\mathbf{U} \sim \mathfrak{C}\mathcal{B}I_m(a, b)$ with $\text{Re}(a) > m - 1$ and $\text{Re}(b) > m - 1$.

2. Then the complex matrix variate beta type II distribution is defined as

$$\mathbf{F} = \begin{cases} \mathbf{B}^{-1/2} \mathbf{A} (\mathbf{B}^{-1/2})', & \text{Definition 1,} \\ \mathbf{A}^{1/2} \mathbf{B}^{-1} (\mathbf{A}^{1/2})', & \text{Definition 2,} \end{cases} \quad (5)$$

Thus under definitions 1 and 2 its density function is denoted as

$$\mathfrak{C}\mathcal{B}II_m(\mathbf{U}; a, b),$$

and given by

$$\frac{1}{\mathfrak{C}\beta_m[a, b]} |\mathbf{F}|^{a-m} |\mathbf{I}_m + \mathbf{F}|^{-(a+b)} (d\mathbf{F}), \quad \mathbf{F} = \mathbf{F}^H > \mathbf{0}. \quad (6)$$

this being denoted as $\mathbf{F} \sim \mathfrak{C}\mathcal{B}II_m(a, b)$ with $\text{Re}(a) > m - 1$ and $\text{Re}(b) > m - 1$.

2.2. Complex hypergeometric functions and invariant polynomials

The following definitions of hypergeometric functions with a matrix argument are based on Constantine [6] and Koev and Edelman [19].

Definition 3. The hypergeometric functions of a matrix argument are given by

$${}_pF_q^{(\alpha)}(a_1, \dots, a_p; b_1, \dots, b_q; \mathbf{X}) = \sum_{t=0}^{\infty} \sum_{\tau} \frac{(a_1)_{\tau}^{(\alpha)} \dots (a_p)_{\tau}^{(\alpha)} C_{\tau}^{(\alpha)}(\mathbf{X})}{(b_1)_{\tau}^{(\alpha)} \dots (b_q)_{\tau}^{(\alpha)} t!}, \quad (7)$$

where \sum_{τ} denotes the summation over all the partitions $\tau = (t_1, \dots, t_m)$, $t_1 \geq \dots \geq t_m \geq 0$, of $t = t_1 + \dots + t_m$, $C_{\tau}^{(\alpha)}(\mathbf{X})$ is the Jack polynomial of \mathbf{X} corresponding to τ and the generalised hypergeometric coefficient $(a)_{\tau}^{(\alpha)}$ is given by

$$(a)_{\tau}^{(\alpha)} = \prod_{j=1}^m (a - (j-1)/(\alpha))_{t_j},$$

where $(a)_t = a(a+1)(a+2)\dots(a+t-1)$, $(a)_0 = 1$ see Koev and Edelman [19]. Here $\mathbf{X} \in \mathbb{C}^{m \times m}$, is a complex symmetric matrix and the parameters a_i, b_j are arbitrary complex numbers.

Other characteristics of the parameters a_i and b_j and the convergence of (7) appear in Muirhead [25, p. 258], Gross and Richards [15] and Ratnarajah et al. [28].

Remark 1. In Definition 3, when $\alpha = 1$ and 2 the complex and real cases are obtained, respectively. In this paper only the complex case is considered. Then, adopting the notation used by James [16], denoting the complex hypergeometric function, generalised hypergeometric coefficient and zonal polynomials as ${}_p\tilde{F}_q^{(\alpha)} \equiv {}_pF_q^{(1)}$, $[a]_{\tau} \equiv (a)_{\tau}^{(1)}$, and $\tilde{C}_{\tau}(\cdot) \equiv C_{\tau}^{(1)}(\cdot)$, respectively

A special case of (7) is

$$\begin{aligned} {}_1\tilde{F}_0(a; \mathbf{X}) &= \sum_{t=0}^{\infty} \sum_{\tau} [a]_{\tau} \frac{\tilde{C}_{\tau}(\mathbf{X})}{t!}, \quad (||\mathbf{X}|| < 1) \\ &= |\mathbf{I}_m - \mathbf{X}|^{-a}, \end{aligned}$$

where $||\mathbf{X}||$ denotes the maximum of the absolute values of the eigenvalues of \mathbf{X} .

From Ratnarajah et al. [28], is known that,

$${}_1\tilde{F}_1(a; c; \mathbf{X}) = \frac{1}{\mathfrak{C}_{\beta_m}[a, c-a]} \times \int_{\mathbf{0} < \mathbf{Y} = \mathbf{Y}^H < \mathbf{I}_m} {}_0\tilde{F}_0(\mathbf{X}\mathbf{Y}) |\mathbf{Y}|^{a-m} |\mathbf{I} - \mathbf{Y}|^{c-a-m} (d\mathbf{Y})$$

and

$${}_2\tilde{F}_1(a, a_1; c; \mathbf{X}) = \frac{1}{\mathfrak{C}_{\beta_m}[a, c-a]} \times \int_{\mathbf{0} < \mathbf{Y} = \mathbf{Y}^H < \mathbf{I}_m} {}_1\tilde{F}_0(a_1; \mathbf{X}\mathbf{Y}) |\mathbf{Y}|^{a-m} |\mathbf{I} - \mathbf{Y}|^{c-a-m} (d\mathbf{Y}). \quad (8)$$

Thus we have

Lemma 2. Let $\mathbf{X} = \mathbf{X}^H \in \mathbb{C}^{m \times m}$, with $\text{Re}(\mathbf{X}) < \mathbf{I}$, $\text{Re}(a) > m-1$, $\text{Re}(c) > m-1$ and $\text{Re}(c-a) > m-1$. Then

$$\begin{aligned} & {}_{p+1}\tilde{F}_{q+1}(a, a_1, \dots, a_p; c, b_1, \dots, b_q; \mathbf{X}) \\ &= \frac{1}{\mathfrak{C}_{\beta_m}[a, c-a]} \int_{\mathbf{0} < \mathbf{Y} = \mathbf{Y}^H < \mathbf{I}_m} {}_p\tilde{F}_q(a_1 \dots a_p; b_1 \dots b_q; \mathbf{X}\mathbf{Y}) |\mathbf{Y}|^{a-m} |\mathbf{I} - \mathbf{Y}|^{c-a-m} (d\mathbf{Y}). \end{aligned}$$

Proof. First, an expansion is applied in terms of complex zonal polynomials

$${}_p\tilde{F}_q(a_1, \dots, a_p; b_1, \dots, b_q; \mathbf{X}\mathbf{Y}) = \sum_{t=0}^{\infty} \sum_{\tau} \frac{[a_1]_{\tau} \dots [a_p]_{\tau}}{[b_1]_{\tau} \dots [b_q]_{\tau}} \frac{\tilde{C}_{\tau}(\mathbf{X}\mathbf{Y})}{t!}.$$

Then, after integrating term by term, see Ratnarajah et al. [28], we have that

$$\begin{aligned}
& \int_{\mathbf{0} < \mathbf{Y} = \mathbf{Y}^H < \mathbf{I}_m} p\tilde{F}_q(a_1 \cdots a_p; b_1 \cdots b_q; \mathbf{X}\mathbf{Y}) |\mathbf{Y}|^{a-m} |\mathbf{I} - \mathbf{Y}|^{c-a-m} (d\mathbf{Y}) \\
&= \sum_{t=0}^{\infty} \sum_{\tau} \frac{[a_1]_{\tau} \cdots [a_p]_{\tau}}{[b_1]_{\tau} \cdots [b_q]_{\tau} t!} \int_{\mathbf{0} < \mathbf{Y} = \mathbf{Y}^H < \mathbf{I}_m} |\mathbf{Y}|^{a-m} |\mathbf{I} - \mathbf{Y}|^{c-a-m} \tilde{C}_{\tau}(\mathbf{X}\mathbf{Y}) (d\mathbf{Y}) \\
&= \mathfrak{C}\beta_m[a, c-a] \sum_{t=0}^{\infty} \sum_{\tau} \frac{[a]_{\tau}}{[c]_{\tau}} \frac{[a_1]_{\tau} \cdots [a_p]_{\tau}}{[b_1]_{\tau} \cdots [b_q]_{\tau} t!} \tilde{C}_{\tau}(\mathbf{X}) \\
&= \mathfrak{C}\beta_m[a, c-a] \tilde{F}_{q+1}(a, a_1 \cdots a_p; c, b_1 \cdots b_q; \mathbf{X})
\end{aligned}$$

and the required result follows. \square

The use of complex zonal polynomials and the hypergeometric function with a matrix argument has only recently been extended; to a large extent this is derived from the work of Koev and Edelman [19]. Koev [18] provided a program in MatLab with a very efficient algorithm for calculating Jack polynomials (in particular complex zonal polynomials) and the complex hypergeometric function with a matrix argument.

This section concludes by establishing the following two properties of a class of homogeneous polynomials $\tilde{C}_{\phi}^{k,\tau}(\mathbf{R}, \mathbf{S})$ of degrees k and t in the eigenvalues of the Hermitian matrices $\mathbf{R}, \mathbf{S} \in \mathfrak{C}^{m \times m}$, respectively, see Davis [8] and Ratnarajah et al. [27]. These properties generalise the incomplete beta function Eq. (61) of Constantine [6]. The first is proposed by Davis [7, Eq. (3.3)] and the second is obtained using the complex versions of Chikuse [3, Eq. (3.33)] and the review version of Chikuse [3, Eq. (3.11)] given in Chikuse and Davis [5, Eq. (2.7)].

Lemma 3. Let $\mathbf{R}, \mathbf{S}, \mathbf{\Omega}, \mathbf{\Xi} \in \mathfrak{C}^{m \times m}$ be Hermitian matrices. Then

$$\begin{aligned}
& \int_{\mathbf{0}}^{\mathbf{\Lambda} = \mathbf{\Lambda}^H} |\mathbf{R}|^{a-m} |\mathbf{I} - \mathbf{R}|^{b-m} \tilde{C}_{\tau}(\mathbf{\Omega}\mathbf{R}) (d\mathbf{R}) \\
&= \mathfrak{C}\beta_m[a, m] |\mathbf{\Lambda}|^a \sum_{k=0}^{\infty} \sum_{\kappa; \phi \in \kappa, \tau} \frac{[a]_{\phi} [-b+m]_{\kappa} \theta_{\phi}^{k,\tau} \tilde{C}_{\phi}^{k,\tau}(\mathbf{\Lambda}, \mathbf{\Omega}\mathbf{\Lambda})}{k! [a+m]_{\phi}}
\end{aligned} \tag{9}$$

and

$$\begin{aligned}
& \int_{\mathbf{0}}^{\mathbf{\Lambda} = \mathbf{\Lambda}^H} |\mathbf{R}|^{a-m} |\mathbf{I} - \mathbf{R}|^{b-m} \tilde{C}_{\phi}^{k,\tau}(\mathbf{\Xi}, \mathbf{\Omega}\mathbf{R}) (d\mathbf{R}) \\
&= \mathfrak{C}\beta_m[a, m] |\mathbf{\Lambda}|^a \sum_{s=0}^{\infty} \sum_{\sigma; \phi_1 \in \sigma, \phi} \frac{[a]_{\phi_1} [-b+m]_{\sigma} \pi_{\sigma, \phi}^{\sigma, \kappa, \tau; \phi_1} \tilde{C}_{\phi_1}^{\sigma, \kappa, \tau}(\mathbf{\Lambda}, \mathbf{\Xi}, \mathbf{\Omega}\mathbf{\Lambda})}{s! [a+m]_{\phi_1}},
\end{aligned} \tag{10}$$

where, from Chikuse and Davis [5], Lemma 2.2(i) and (ii), respectively it is known that

$$\theta_{\phi}^{k,\tau} = \frac{\tilde{C}_{\phi}^{k,\tau}(\mathbf{I}, \mathbf{I})}{\tilde{C}_{\phi}(\mathbf{I})} \quad \text{and} \quad \pi_{\sigma, \phi}^{\sigma, \kappa, \tau; \phi_1} = \sum_{\phi'_1 \equiv \phi_1} \gamma_{\sigma; \phi'_1}^{\sigma, \kappa, \tau; \phi_1} \bar{\alpha}_{\phi}^{\sigma^*, \kappa, \tau; \phi'_1}.$$

3. Bimatrix variate generalised beta type I distribution

Let \mathbf{A}, \mathbf{B} and \mathbf{C} be independent complex random matrices, such that $\mathbf{A} \sim \mathfrak{C}\mathcal{G}_m(a, \mathbf{I}_m)$, $\mathbf{B} \sim \mathfrak{C}\mathcal{G}_m(b, \mathbf{I}_m)$ and $\mathbf{C} \sim \mathfrak{C}\mathcal{G}_m(c, \mathbf{I}_m)$ with $\text{Re}(a) > m-1$, $\text{Re}(b) > m-1$ and $\text{Re}(c) > m-1$ and let us define

$$\mathbf{U}_1 = (\mathbf{A} + \mathbf{C})^{-1/2} \mathbf{A} (\mathbf{A} + \mathbf{C})^{-1/2} \quad \text{and} \quad \mathbf{U}_2 = (\mathbf{B} + \mathbf{C})^{-1/2} \mathbf{B} (\mathbf{B} + \mathbf{C})^{-1/2}. \tag{11}$$

Of course, $\mathbf{U}_1 \sim \mathcal{CB}I_m(a, c)$ and $\mathbf{U}_2 \sim \mathcal{CB}I_m(b, c)$. However, they are correlated such that the distribution of $\mathbb{U} = (\mathbf{U}_1; \mathbf{U}_2)' \in \mathbb{C}^{2m \times m}$ can be termed a complex bimatrix variate generalised beta type I distribution, denoted as

$$\mathbb{U} \sim \mathcal{CBGBI}_{2m \times m}(a, b, c).$$

Theorem 1. Assume that $\mathbb{U} \sim \mathcal{CBGBI}_{2m \times m}(a, b, c)$. Then its density function is

$$\frac{|\mathbf{U}_1|^{a-m} |\mathbf{U}_2|^{b-m} |\mathbf{I}_m - \mathbf{U}_1|^{b+c-m} |\mathbf{I}_m - \mathbf{U}_2|^{a+c-m}}{\mathcal{C}\beta_m^*[a, b, c] |\mathbf{I}_m - \mathbf{U}_1 \mathbf{U}_2|^{a+b+c}} (d\mathbb{U}), \quad (12)$$

$\mathbf{0} < \mathbf{U}_1 = \mathbf{U}_1^H < \mathbf{I}_m$, $\mathbf{0} < \mathbf{U}_2 = \mathbf{U}_2^H < \mathbf{I}_m$, where the measure

$$(d\mathbb{U}) = (d\mathbf{U}_1) \wedge (d\mathbf{U}_2),$$

and

$$\mathcal{C}\beta_m^*[a, b, c] = \frac{\mathcal{C}\Gamma_m[a] \mathcal{C}\Gamma_m[b] \mathcal{C}\Gamma_m[c]}{\mathcal{C}\Gamma_m[a+b+c]}.$$

and $\text{Re}(a) > m-1$, $\text{Re}(b) > m-1$ and $\text{Re}(c) > m-1$.

Proof. The joint density of \mathbf{A} , \mathbf{B} and \mathbf{C} is

$$\frac{|\mathbf{A}|^{a-m} |\mathbf{B}|^{b-m} |\mathbf{C}|^{c-m}}{\mathcal{C}\Gamma_m[a] \mathcal{C}\Gamma_m[b] \mathcal{C}\Gamma_m[c]} \text{etr}(-(\mathbf{A} + \mathbf{B} + \mathbf{C})) (d\mathbf{A})(d\mathbf{B})(d\mathbf{C}).$$

By effecting the change of variable (11), and taking into account Mathai [21, Theorems 3.5 and 3.8, pp. 183 and 190], respectively, we have

$$(d\mathbf{A})(d\mathbf{B})(d\mathbf{C}) = |\mathbf{C}|^{2m} |\mathbf{I}_m - \mathbf{U}_1|^{-2m} |\mathbf{I}_m - \mathbf{U}_2|^{-2m} (d\mathbf{U}_1)(d\mathbf{U}_2)(d\mathbf{C}).$$

The joint density of \mathbf{U}_1 , \mathbf{U}_2 and \mathbf{C} is

$$\frac{|\mathbf{U}_1|^{a-m} |\mathbf{U}_2|^{b-m}}{\mathcal{C}\Gamma_m[a] \mathcal{C}\Gamma_m[b] \mathcal{C}\Gamma_m[c] |\mathbf{I}_m - \mathbf{U}_1|^{a+m} |\mathbf{I}_m - \mathbf{U}_2|^{b+m}} |\mathbf{C}|^{a+b+c-m} \\ \times \text{etr} \left[-(\mathbf{I}_m - \mathbf{U}_2)^{-1} (\mathbf{I}_m - \mathbf{U}_1 \mathbf{U}_2) (\mathbf{I}_m - \mathbf{U}_1)^{-1} \mathbf{C} \right] (d\mathbf{C})(d\mathbf{U}_1)(d\mathbf{U}_2).$$

Integrating with respect to \mathbf{C} using

$$\int_{\mathbf{C}=\mathbf{C}^H > \mathbf{0}} |\mathbf{C}|^{a+b+c-m} \text{etr} \left[-(\mathbf{I}_m - \mathbf{U}_2)^{-1} (\mathbf{I}_m - \mathbf{U}_1 \mathbf{U}_2) (\mathbf{I}_m - \mathbf{U}_1)^{-1} \mathbf{C} \right] (d\mathbf{C}) \\ = \mathcal{C}\Gamma[a+b+c] \frac{|\mathbf{I}_m - \mathbf{U}_1|^{a+b+c} |\mathbf{I}_m - \mathbf{U}_2|^{a+b+c}}{|\mathbf{I}_m - \mathbf{U}_1 \mathbf{U}_2|^{a+b+c}}$$

(from (2)) gives the stated marginal density function for $(\mathbf{U}_1; \mathbf{U}_2)'$. \square

As in the real case [10], the joint density (12) can be represented as a mixture. Let us first note that

$$|\mathbf{I}_m - \mathbf{U}_1 \mathbf{U}_2|^{-(a+b+c)} = {}_1\tilde{F}_0(a+b+c; \mathbf{U}_1 \mathbf{U}_2) \\ = \sum_{t=0}^{\infty} \sum_{\tau} [a+b+c]_{\tau} \frac{\tilde{\mathcal{C}}_{\tau}(\mathbf{U}_1 \mathbf{U}_2)}{t!}.$$

By substituting in (12) it is obtained that the joint density function of $(\mathbf{U}_1; \mathbf{U}_2)'$ is

$$\sum_{t=0}^{\infty} \sum_{\tau} \frac{[a+b+c]_{\tau}}{\mathfrak{C}\beta_m^*[a,b,c]} |\mathbf{U}_1|^{a-m} |\mathbf{U}_2|^{b-m} |\mathbf{I}_m - \mathbf{U}_1|^{b+c-m} \times |\mathbf{I}_m - \mathbf{U}_2|^{a+c-m} \frac{\tilde{\mathfrak{C}}_{\tau}(\mathbf{U}_1 \mathbf{U}_2)}{t!}. \quad (13)$$

Moreover

$$\begin{aligned} & \sum_{t=0}^{\infty} \sum_{\tau} \frac{[a+b+c]_{\tau} \mathfrak{C}\Gamma_m[b+c] \mathfrak{C}\Gamma_m[a+c]}{\mathfrak{C}\Gamma_m[a+b+c] \mathfrak{C}\Gamma_m[c]} \mathfrak{C}\mathcal{B}I_m(\mathbf{U}_1; a, b+c) \\ & \times \mathfrak{C}\mathcal{B}I_m(\mathbf{U}_2; b, a+c) \frac{\tilde{\mathfrak{C}}_{\tau}(\mathbf{U}_1 \mathbf{U}_2)}{t!}. \end{aligned}$$

4. Bimatrix variate generalised beta type II distribution

Let \mathbf{A}, \mathbf{B} and $\mathbf{C} \in \mathbb{C}^{m \times m}$ be independent, where $\mathbf{A} \sim \mathfrak{C}\mathcal{G}_m(a, \mathbf{I}_m)$, $\mathbf{B} \sim \mathfrak{C}\mathcal{G}_m(b, \mathbf{I}_m)$ and $\mathbf{C} \sim \mathfrak{C}\mathcal{G}_m(c, \mathbf{I}_m)$ with $\text{Re}(a) > m-1$, $\text{Re}(b) > m-1$ and $\text{Re}(c) > m-1$ and let us define

$$\mathbf{F}_1 = \mathbf{C}^{-1/2} \mathbf{A} \mathbf{C}^{-1/2} \quad \text{and} \quad \mathbf{F}_2 = \mathbf{C}^{-1/2} \mathbf{B} \mathbf{C}^{-1/2}. \quad (14)$$

Clearly, $\mathbf{F}_1 \sim \mathfrak{C}\mathcal{B}I_m(a, c)$ and $\mathbf{F}_2 \sim \mathfrak{C}\mathcal{B}I_m(b, c)$. But they are correlated and so the distribution of

$\mathbb{F} = (\mathbf{F}_1; \mathbf{F}_2)' \in \mathbb{C}^{2m \times m}$ can be termed a complex bimatrix variate generalised beta type II distribution, which is denoted as $\mathbb{F} \sim \mathfrak{C}\mathcal{B}GI_{2m \times m}(a, b, c)$.

Theorem 2. Assume that $\mathbb{F} \sim \mathfrak{C}\mathcal{B}GI_{2m \times m}(a, b, c)$. Then its density function is

$$\frac{|\mathbf{F}_1|^{a-m} |\mathbf{F}_2|^{b-m}}{\mathfrak{C}\beta_m^*[a,b,c] |\mathbf{I}_m + \mathbf{F}_1 + \mathbf{F}_2|^{a+b+c}} (d\mathbb{F}), \quad (15)$$

$\mathbf{F}_1 = \mathbf{F}_1^H > \mathbf{0}, \mathbf{F}_2 = \mathbf{F}_2^H > \mathbf{0}$, where the measure

$$(d\mathbb{F}) = (d\mathbf{F}_1) \wedge (d\mathbf{F}_2).$$

and $\text{Re}(a) > m-1$, $\text{Re}(b) > m-1$ and $\text{Re}(c) > m-1$.

Proof. As an alternative to proceeding as in Theorem 1, let us recall that if $\mathbf{U} \sim \mathfrak{C}\mathcal{B}I_m(a, b)$, then $(\mathbf{I}_m - \mathbf{U})^{-1} - \mathbf{I}_m \sim \mathfrak{C}\mathcal{B}I_m(a, b)$, see Srivastava and Khatri [29] and Díaz-García and Gutiérrez-Jáimez [9]. Then

$$\mathbb{F} = \begin{pmatrix} \mathbf{F}_1 \\ \mathbf{F}_2 \end{pmatrix} = \begin{pmatrix} (\mathbf{I}_m - \mathbf{U}_1)^{-1} - \mathbf{I}_m \\ (\mathbf{I}_m - \mathbf{U}_2)^{-1} - \mathbf{I}_m \end{pmatrix},$$

with the Jacobian given by (see [21, Theorem 3.8, p. 190])

$$(d\mathbf{U}_1)(d\mathbf{U}_2) = |\mathbf{I}_m + \mathbf{F}_1|^{-2m} |\mathbf{I}_m + \mathbf{F}_2|^{-2m} (d\mathbf{F}_1)(d\mathbf{F}_2).$$

Also, note that

$$\begin{aligned} \mathbf{I}_m - (\mathbf{I}_m + \mathbf{F}_1)^{-1} &= (\mathbf{I}_m + \mathbf{F}_1)^{-1} ((\mathbf{I}_m + \mathbf{F}_1) - \mathbf{I}_m) = (\mathbf{I}_m + \mathbf{F}_1)^{-1} \mathbf{F}_1 \\ \mathbf{I}_m - (\mathbf{I}_m + \mathbf{F}_2)^{-1} &= (\mathbf{I}_m + \mathbf{F}_2)^{-1} \mathbf{F}_2, \end{aligned}$$

Then the joint density of $(\mathbf{F}_1; \mathbf{F}_2)'$ is

$$\frac{|\mathbf{F}_1|^{a-m} |\mathbf{F}_2|^{b-m} |\mathbf{I}_m + \mathbf{F}_1|^{-(a+b+c)} |\mathbf{I}_m + \mathbf{F}_2|^{-(a+b+c)}}{\mathfrak{C}\beta_m^*[a,b,c] |\mathbf{I}_m - (\mathbf{I}_m + \mathbf{F}_1)^{-1} \mathbf{F}_1 \mathbf{F}_2 (\mathbf{I}_m + \mathbf{F}_2)^{-1}|^{a+b+c}} (d\mathbb{F}).$$

The desired result then follows, noting that

$$\frac{|\mathbf{I}_m + \mathbf{F}_1|^{-1} |\mathbf{I}_m + \mathbf{F}_2|^{-1}}{|\mathbf{I}_m - (\mathbf{I}_m + \mathbf{F}_1)^{-1} \mathbf{F}_1 \mathbf{F}_2 (\mathbf{I}_m + \mathbf{F}_2)^{-1}|} = |\mathbf{I}_m + \mathbf{F}_1 + \mathbf{F}_2|^{-1}. \quad \square$$

Other properties of the distribution $\mathfrak{CBGBl}_{2m \times m}(a, b, c)$ can be found in a similar way.

5. Properties

In this section, several basic properties and eigenvalue distributions are studied.

5.1. Basic properties

As direct consequences of Lemma 2 the following basic properties are derived: the moments $E(|\mathbf{U}_1|^r |\mathbf{U}_2|^s)$, the distributions of the product $\mathbf{Z} = \mathbf{U}_2^{1/2} \mathbf{U}_1 \mathbf{U}_2^{1/2}$ and the inverse $(\mathbf{U}_1^{-1}; \mathbf{U}_2^{-1})$. Their proofs are similar to those given for the real case, see Díaz-García and Gutiérrez-Jáimez [10].

Theorem 3. Assume that $(\mathbf{U}_1; \mathbf{U}_2) \sim \mathfrak{CBGBl}_{2m \times m}(a, b, c)$ then

$$E(|\mathbf{U}_1|^r |\mathbf{U}_2|^s) = \frac{\mathfrak{C}\beta_m[a + r, b + c] \mathfrak{C}\beta_m[b + s, a + c]}{\mathfrak{C}\beta_m^*[a, b, c]} \times {}_3\tilde{F}_2(a + r, b + s, a + b + c; a + b + c + r, a + b + c + s; \mathbf{I}_m),$$

with $\text{Re}(b + r) > m - 1$, and $\text{Re}(a + c) > m - 1$.

Theorem 4. Consider $(\mathbf{U}_1; \mathbf{U}_2) \sim \mathfrak{CBGBl}_{2m \times m}(a, b, c)$. Then the density function of $\mathbf{Z} = \mathbf{Z}^H = \mathbf{U}_2^{1/2} \mathbf{U}_1 \mathbf{U}_2^{1/2} \in \mathfrak{C}^{m \times m}$ is

$$\frac{\mathfrak{C}\beta_m[a + c, b + c] |\mathbf{Z}|^{a-m} |\mathbf{I}_m - \mathbf{Z}|^{c-m}}{\mathfrak{C}\beta_m^*[a, b, c]} {}_2\tilde{F}_1(a + c, a + c; a + b + 2c; \mathbf{I}_m - \mathbf{Z}) (d\mathbf{Z})$$

and

$$E(|\mathbf{Z}|^r) = \frac{\mathfrak{C}\beta_m[a + c, b + c] \mathfrak{C}\beta_m[a + r, c]}{\mathfrak{C}\beta_m^*[a, b, c]} {}_3\tilde{F}_2(c, a + c, a + c; a + c + r, a + b + 2c; \mathbf{I}_m),$$

with $0 < \text{Re}(\mathbf{Z}) < \mathbf{I}_m$, $\text{Re}(a + b) > m - 1$ and $\text{Re}(b + c) > m - 1$.

Theorem 5. Let $(\mathbf{U}_1; \mathbf{U}_2)' \sim \mathfrak{CBGBl}_{2m \times m}(a, b, c)$. Then the density function of $\mathbb{V} = (\mathbf{V}_1; \mathbf{V}_2)' = (\mathbf{U}_1^{-1}; \mathbf{U}_2^{-1})' \in \mathfrak{C}^{2m \times m}$ is

$$\frac{|\mathbf{V}_1|^{-a-m} |\mathbf{V}_2|^{-b-m} |\mathbf{I}_m - \mathbf{V}_1^{-1}|^{b+c-m} |\mathbf{I}_m - \mathbf{V}_2^{-1}|^{a+c-m}}{\mathfrak{C}\beta_m^*[a, b, c] |\mathbf{I}_m - (\mathbf{V}_1 \mathbf{V}_2)^{-1}|^{a+b+c}} (d\mathbb{V}),$$

$0 < \mathbf{V}_1 = \mathbf{V}_1^H < \mathbf{I}_m$, $0 < \mathbf{V}_2 = \mathbf{V}_2^H < \mathbf{I}_m$, where the measure

$$(d\mathbb{V}) = (d\mathbf{V}_1) \wedge (d\mathbf{V}_2).$$

and $\text{Re}(a) > m - 1$, $\text{Re}(b) > m - 1$ and $\text{Re}(c) > m - 1$.

5.2. Joint eigenvalue distribution

Many test statistics in multivariate analysis are functions of the eigenvalues or of the maximum eigenvalue. In these final two subsections, we find the eigenvalues distribution. The following result is needed, see James [16].

Lemma 4. If $f_A(\mathbf{A}) (d\mathbf{A})$ is the density function of a Hermitian matrix variate $\mathbf{A} \in \mathbb{C}^{m \times m}$, then the distribution of the diagonal matrix $\mathbf{\Lambda} = \text{diag}(\lambda_1, \dots, \lambda_m)$, $\lambda_1 > \dots > \lambda_m > 0$, of the eigenvalues of \mathbf{A} , where $\mathbf{A} = \mathbf{G}\mathbf{\Lambda}\mathbf{G}^H$, $\mathbf{G} \in \mathcal{U}(m)$, is the eigendecomposition of \mathbf{A} , is

$$f_{\mathbf{\Lambda}}(\mathbf{\Lambda}) = \frac{\pi^{m(m-1)}}{\mathfrak{C}\Gamma_m[m]} \prod_{r < s} (\lambda_r - \lambda_s)^2 \int_{\mathcal{U}(m)} f_{\mathbf{A}}(\mathbf{G}\mathbf{\Lambda}\mathbf{G}^H)(d\mathbf{G}),$$

where $(d\mathbf{G})$ is the invariant measure on the unitary group $\mathcal{U}(m)$ normalised, given as

$$(d\mathbf{G}) = \frac{\mathfrak{C}\Gamma_m[m]}{2^m \pi^{m^2}} (\mathbf{G} d\mathbf{G}^H), \quad \text{such that} \quad \int_{\mathcal{U}(m)} (d\mathbf{G}) = 1. \quad (16)$$

Theorem 6. Assume that $(\mathbf{U}_1; \mathbf{U}_2) \sim \mathfrak{CBGBI}_{2m \times m}(a, b, c)$ and let

$$\mathbb{U} = \begin{pmatrix} \mathbf{U}_1 \\ \mathbf{U}_2 \end{pmatrix} = \begin{pmatrix} \mathbf{E}\mathbf{D}_{\lambda}\mathbf{E}^H \\ \mathbf{G}\mathbf{D}_{\delta}\mathbf{G}^H \end{pmatrix}. \quad (17)$$

The spectral decomposition of \mathbf{U}_1 and \mathbf{U}_2 , with $\mathbf{E}, \mathbf{G} \in \mathcal{U}(m)$ and $\mathbf{D}_{\lambda} = \text{diag}(\lambda_1, \dots, \lambda_m)$, $1 > \lambda_1 > \dots > \lambda_m > 0$ and $\mathbf{D}_{\delta} = \text{diag}(\delta_1, \dots, \delta_m)$, $1 > \delta_1 > \dots > \delta_m > 0$. Then the joint density function of $\lambda_1, \dots, \lambda_m, \delta_1, \dots, \delta_m$ is

$$\begin{aligned} & \frac{\pi^{2m(m-1)}}{(\mathfrak{C}\Gamma_m[m])^2 \mathfrak{C}\beta^*[a, b, c]} \prod_{r=1}^m (\lambda_r^{a-m} (1 - \lambda_r)^{b+c-m}) \prod_{e=1}^m (\delta_e^{b-m} (1 - \delta_e)^{a+c-m}) \\ & \times \prod_{r < s} (\lambda_r - \lambda_s)^2 \prod_{e < f} (\delta_e - \delta_f)^2 \sum_{k=0}^{\infty} \sum_{\kappa} \frac{[a + b + c]_{\kappa}}{k!} \frac{\tilde{\mathfrak{C}}_{\kappa}(\mathbf{D}_{\lambda}) \tilde{\mathfrak{C}}_{\kappa}(\mathbf{D}_{\delta})}{\tilde{\mathfrak{C}}_{\kappa}(\mathbf{I}_m)}, \end{aligned}$$

with $\text{Re}(a) > m - 1$, $\text{Re}(b) > m - 1$ and $\text{Re}(c) > m - 1$.

Proof. The desired result follows by making the transformation (17), from Lemma 4, Eq. (16) and observing that, see Chikuse [4]

$$\int_{\mathbf{G} \in \mathcal{U}(m)} \int_{\mathbf{E} \in \mathcal{U}(m)} \tilde{\mathfrak{C}}_{\kappa}(\mathbf{E}\mathbf{D}_{\lambda}\mathbf{E}^H \mathbf{G}\mathbf{D}_{\delta}\mathbf{G}^H)(d\mathbf{E})(d\mathbf{G}) = \frac{\tilde{\mathfrak{C}}_{\kappa}(\mathbf{D}_{\lambda}) \tilde{\mathfrak{C}}_{\kappa}(\mathbf{D}_{\delta})}{\tilde{\mathfrak{C}}_{\kappa}(\mathbf{I})}. \quad \square$$

5.3. Joint distribution of λ_{\max} and δ_{\max}

In this subsection, we derive the distribution of the largest eigenvalues, λ_{\max} and δ_{\max} of a complex bimatrix variate beta type I matrix. With this aim, let us consider the following theorem.

Theorem 7. Assume that $(\mathbf{U}_1; \mathbf{U}_2) \sim \mathfrak{CBGBI}_{2m \times m}(a, b, c)$ and let $\mathbf{\Delta}_1, \mathbf{\Delta}_2 \in \mathbb{C}^{m \times m}$ be Hermitian positive definite matrices, $\mathbf{0} < \mathbf{\Delta}_1, \mathbf{\Delta}_2 < \mathbf{I}$. Then the probability $P(\mathbf{U}_1 < \mathbf{\Delta}_1, \mathbf{U}_2 < \mathbf{\Delta}_2)$ is given by

$$\frac{\mathfrak{C}\beta_m[a, m] \mathfrak{C}\beta_m[b, m]}{\mathfrak{C}\beta_m^*[a, b, c]} |\mathbf{\Delta}_1|^a |\mathbf{\Delta}_2|^b \sum_{k, t, s=0}^{\infty} \sum_{\kappa, \tau, \sigma; \phi_1 \in \kappa, \tau; \phi_2 \in \sigma^*, \phi_1^*} \frac{[a]_{\phi_2} [b]_{\phi_1} [a + b + c]_{\kappa}}{k! t! s!}$$

$$\times \frac{[-(a+c)+m]_{\tau}[-(b+c)+m]_{\sigma}}{[a+m]_{\phi_2}[b+m]_{\phi_1}} \theta_{\phi_1}^{\kappa,\tau} \pi_{\sigma,\phi_1}^{\sigma,\tau,\kappa;\phi_2} \tilde{C}_{\phi_2}^{\sigma,\tau,\kappa}(\Delta_1, \Delta_2, \Delta_1 \Delta_2),$$

where

$$\sum_{\kappa,\tau,\sigma;\phi_1 \in \kappa.\tau;\phi_2 \in \sigma^*.\phi_1^*} = \sum_{\kappa,\tau,\sigma} \sum_{\phi_1 \in \kappa.\tau} \sum_{\phi_2 \in \sigma^*.\phi_1^*}$$

and $\operatorname{Re}(a) > m-1$, $\operatorname{Re}(b) > m-1$ and $\operatorname{Re}(c) > m-1$.

Proof. From (12) the probability $P(\mathbf{U}_1 < \Delta_1, \mathbf{U}_2 < \Delta_2)$ is

$$\int_0^{\Delta_1} \int_0^{\Delta_2} \frac{|\mathbf{U}_1|^{a-m} |\mathbf{U}_2|^{b-m} |\mathbf{I}_m - \mathbf{U}_1|^{b+c-m} |\mathbf{I}_m - \mathbf{U}_2|^{a+c-m}}{\mathfrak{C}\beta_m^*[a, b, c] |\mathbf{I}_m - \mathbf{U}_1 \mathbf{U}_2|^{a+b+c}} (d\mathbf{U}_1)(d\mathbf{U}). \quad (18)$$

By rewriting (18) as in (13), and integrating with respect to \mathbf{U}_2 using (9), it is obtained that

$$\frac{\mathfrak{C}\beta_m[b, m]}{\mathfrak{C}\beta_m^*[a, b, c]} |\Delta_1|^a \sum_{k,t=0}^{\infty} \sum_{\kappa,\tau;\phi_1 \in \kappa.\tau} \frac{[b]_{\phi_1} [a+b+c]_{\kappa} [-(a+c)+m]_{\tau} \theta_{\phi_1}^{\kappa,\tau}}{k! t! [b+m]_{\phi_1}} \\ \times \int_0^{\Delta_1} |\mathbf{U}_1|^{a-m} |\mathbf{I}_m - \mathbf{U}_1|^{b+c-m} \tilde{C}_{\phi_1}^{\tau,\kappa}(\Delta_2, \Delta_2 \mathbf{U}_1) (d\mathbf{U}_1).$$

The desired result follows by integrating with respect to \mathbf{U}_1 using (10). \square

The following result is obtained from Theorem 7.

Corollary 1. Let $(\mathbf{U}_1; \mathbf{U}_2) \sim \mathfrak{CBGBI}_{2m \times m}(a, b, c)$, $\operatorname{Re}(a) > m-1$, $\operatorname{Re}(b) > m-1$ and $\operatorname{Re}(c) > m-1$. If λ_{\max} and δ_{\max} are the largest eigenvalues of \mathbf{U}_1 and \mathbf{U}_2 , respectively, then their joint distribution function, $P(\lambda_{\max} < x, \delta_{\max} < y)$ is given by

$$\frac{\mathfrak{C}\beta_m[a, m] \mathfrak{C}\beta_m[b, m]}{\mathfrak{C}\beta_m^*[a, b, c]} \sum_{k,t,s=0}^{\infty} \sum_{\kappa,\tau,\sigma;\phi_1 \in \kappa.\tau;\phi_2 \in \sigma^*.\phi_1^*} \frac{x^{am+s+k} y^{bm+t+k} [a]_{\phi_2} [b]_{\phi_1}}{k! t! s!} \\ \times \frac{[-(a+c)+m]_{\tau}[-(b+c)+m]_{\sigma}}{[a+b+c]_{\kappa} [a+m]_{\phi_2} [b+m]_{\phi_1}} \theta_{\phi_1}^{\kappa,\tau} \pi_{\sigma,\phi_1}^{\sigma,\tau,\kappa;\phi_2} \theta_{\phi_2}^{\kappa,\tau,\sigma} \tilde{C}_{\phi_2}^{\sigma,\tau,\kappa}(\mathbf{I}).$$

Proof. The inequalities $\lambda_{\max} < x$ and $\delta_{\max} < y$ are equivalent to $\mathbf{U}_1 < x\mathbf{I}$ and $\mathbf{U}_2 < y\mathbf{I}$. Therefore, the result follows by letting $\Delta_1 = x\mathbf{I}$ and $\Delta_2 = y\mathbf{I}$ in Theorem 7, and taking into account that from Davis [7, Eqs. (2.1) and (2.7)]

$$\tilde{C}_{\phi_2}^{\sigma,\tau,\kappa}(x\mathbf{I}, y\mathbf{I}, xy\mathbf{I}) = x^{s+k} y^{t+k} \tilde{C}_{\phi_2}^{\sigma,\tau,\kappa}(\mathbf{I}). \quad \square$$

6. Conclusions

Random matrices play a prominent role because of their deep mathematical structure. They have arisen in a number of fields (statistics, graph theory, stochastic linear algebra, physics, signal processing, etc.), often independently. In particular, statisticians are interested in studying the Normal (Gaussian), Wishart, beta type I, and circular random matrices, which, from the point of view of *random matrix theory* are termed Hermite, Laguerre, Jacobi, and Fourier ensembles, Metha [23] and Edelman and Rao [12].

The results obtained in the present work can be considered as a generalisation of the Jacobi ensemble for the case in which there are two correlated Jacobi ensembles. For all their potential applications, see Edelman and Sutton [13].

Another potential use appears in the context of complex shape theory, see Micheas et al. [24], more specifically, in the approach known as affine shape or configuration densities, see Caro-Lopera et al. [1].

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